Interior Schauder estimates

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Before proving the Schauder estimates, we want to prove the following abstract lemma:

Lemma 1 (Simple Abstract Lemma). *Let $B_R(x_0)$ be any ball in $\mathbb{R}^n$, $k \in \mathbb{R}$, $\theta \in (0,1)$, $\gamma \in (0,\infty)$, and $\nu \in (0,1]$. Let S be a nonnegative function on the class of convex open subsets of $B_R(x_0)$ and suppose that S is subadditive, i.e. given a finite collection of convex subsets $A, A_1, A_2, \ldots, A_N$ of $B_R(x_0)$,*

$$S(A) \leq \sum_{j=1}^N S(A_j) \text{ whenever } A \subseteq \bigcup_{j=1}^N A_j.$$

Then there is an $\delta = \delta(n,k,\theta)$ such that if

$$\rho^k S(B_{\theta\rho}(y)) \leq \delta \rho^k S(B_\rho(y)) + \gamma \tag{1}$$

whenever $B_\rho(y) \subseteq B_R(x_0)$ and $\rho \leq \nu R$, then

$$R^k S(B_{\theta R}(x_0)) \leq C\gamma$$

for some constant $C = C(n,k,\theta,\nu)$.

We will use the Simple Abstract Lemma in the special case S be a seminorm of a solution to an elliptic equation as a trick for obtaining estimates. For instance, in the proof of the Schauder estimates, we will let $S(A) = [D^2u]_{\mu;A}$ for all convex open sets A in $B_R(x)$. We will then obtain

$$R^{2+\mu}[D^2u]_{\mu;B_{R/2}(y)} \leq \delta R^{2+\mu}[D^2u]_{\mu;B_R(y)} + C\left(\sup_{B_R(x_0)} |u| + \|f\|'_{C^{0,\mu}(B_R(x_0))}\right)$$

for a constant $C \in (0,\infty)$. One might regard this as a bad estimate because we are bounding $[D^2u]_{\mu;B_{\theta\rho}(y)}$ in terms of precisely the thing we want to bound, Hölder coefficients of D^2u, and on larger balls no less. However, the Simple Abstract Lemma allows us to absorb the $[D^2u]_{\mu;B_\rho(y)}$ term into the left-hand side to conclude that

$$R^{2+\mu}[D^2u]_{\mu;B_{R/2}(y)} \leq C\left(\sup_{B_R(x_0)} |u| + \|f\|'_{C^{0,\mu}(B_R(x_0))}\right)$$

for a constant $C \in (0,\infty)$ as desired.

Proof of the Simple Abstract Lemma. Let

$$Q = \sup_{B_\rho(y) \subset B_R(x), \rho \leq \nu R} \rho^k S(B_{\theta\rho}(y)).$$

By (1),

$$(\theta\rho)^k S(B_{\theta^2\rho}(y)) \leq \delta Q + \gamma \tag{2}$$

whenever $B_\rho(y) \subset B_R(x)$ with $\rho \leq \nu R$. Take an arbitrary ball $B_\rho(y) \subset B_R(x)$ with $\rho \leq \nu R$ and cover $B_{\theta\rho}(y)$ by a finite collection of smaller open balls $B_{\theta^2(1-\theta)\rho}(z_j)$, $j = 1, \ldots, N$, with $z_j \in B_{\theta\rho}(y)$ and $N \leq C$ for some constant $C = C(n, \theta) \in (0, \infty)$. Since $B_{(1-\theta)\rho}(z_j) \subset B_R(x)$, we can replace $B_\rho(y)$ with $B_{(1-\theta)}(z_j)$ in (2) and sum over j to obtain

$$\begin{aligned} \rho^k S(B_{\theta\rho}(y)) &\leq \sum_{j=1}^N \rho^k S(B_{\theta^2(1-\theta)\rho}(z_j)) && \text{(by subadditivity of } S) \\ &\leq N(\theta(1-\theta)\rho)^{-k}(\delta Q + \gamma) && \text{(by (2))} \\ &\leq C(\delta Q + \gamma) \end{aligned}$$

for some constant $C = C(n, k, \theta) \in (0, \infty)$. Since $B_\rho(y)$ is arbitrary,

$$Q \leq C(\delta Q + \gamma). \tag{3}$$

Choosing δ such that $C\delta < 1/2$ in (3), we obtain

$$Q \leq 2C\gamma;$$

that is,

$$\rho^k S(B_{\theta\rho}(y)) \leq 2C\gamma \tag{4}$$

whenever $B_\rho(y) \subset B_R(x)$ with $\rho \leq \nu R$.

Now without the restriction $\rho \leq \nu R$ (i.e. the case where $\nu = 1$) we would be done as would could simply choose $B_\rho(y) = B_R(x)$. We cover $B_{\theta R}(x)$ by a finite collection $\{B_{\theta\nu R}(y_j)\}_{j=1,\ldots,N'}$ of open balls such that $B_{\nu R}(y_j) \subset B_R(x)$ and $N' \leq C$ for some constant $C = C(n, \theta, \nu) \in (0, \infty)$. By replacing $B_\rho(y)$ with $B_{\nu R}(y_j)$ in (4) and summing over j using the subadditivity of S,

$$R^k S(B_{\theta R}(x)) \leq \sum_{j=1}^N R^k S(B_{\theta\nu R}(y_j)) \leq C\gamma$$

for $C = C(n, k, \theta, \nu) \in (0, \infty)$. □

Now we want to prove the following interior Schauder estimate.

Lemma 2. *Let $\mu \in (0, 1)$. Consider a ball $B_R(x_0)$ in $\mathbb{R}^n$. Suppose $u \in C^{2,\mu}(\overline{B_R(x_0)})$ solves the elliptic equation*

$$a^{ij} D_{ij} u + b^i D_i u + cu = f \text{ in } B_R(x_0), \tag{5}$$

where $a^{ij}, b^i, c : C^{0,\mu}(\overline{B_R(x_0)})$ are coefficients and $f \in C^{0,\mu}(\overline{B_R(x_0)})$. Assume the coefficients satisfy the bounds

$$\lambda|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \text{ for } x \in B_R(x_0),\, \xi \in \mathbb{R}^n,$$

$$\sum_{i,j=1}^n \|a^{ij}\|'_{C^{0,\mu}(B_R(x_0))} + \sum_{i=1}^n R\|b^i\|'_{C^{0,\mu}(B_R(x_0))} + R^2\|c\|'_{C^{0,\mu}(B_R(x_0))} \leq \beta,$$

for some constants $\lambda, \Lambda, \beta \in (0, \infty)$ *such that* $0 < \lambda \leq \Lambda$ *and* $f \in C^{0,\mu}(\overline{B_R(x_0)})$. *Then*

$$|u|'_{2,\mu;B_{R/2}(x_0)} \leq C \left(|u|_{0;B_R(x_0)} + R^2 |f|'_{0,\mu;B_R(x_0)} \right) \tag{6}$$

for some constant $C = C(n, \mu, \lambda, \Lambda, \beta) \in (0, \infty)$.

What the Schauder estimate roughly states is that the $C^{2,\mu}$ norm of u is bounded in terms of the supremum of u and norm of f. This is what one would expect. Consider for instance that for a solution u to an elliptic equation on an interval in $\mathbb{R}$, we can explicitly express u in terms of its values at the boundary of the interval and f.

Remarks:

(1) We need $0 < \mu < 1$. The Schauder estimate is not true in general with $\mu = 1$. This basically follows from the fact that there exists an $f \in C^{0,1}(B_1(0))$ and a solution $u \in C^2(B_1(0))$ to $\Delta u = f$ in $B_1(0)$ such that $u \notin C^{2,1}$ in any neighborhood of the origin (see the example sheet). Now if there were a $C^{2,1}$ Schauder estimate, then we could use convolution to approximate u and f by smooth functions u_k and f_k such that $\Delta u_k = f_k$ in $B_1(0)$ and $u_k \to u$ in $C^2(B_1(0))$ and $f_k \to f$ uniformly on compact subsets of $B_1(0)$ and the $C^{2,1}$ Schauder estimate and Arzela-Ascoli would imply that $u \in C^{2,1}(B_1(0))$! The idea here is that we will later use the Schauder estimates to build up a theory for compactness, existence, and regularity of solutions to elliptic equations and since the $C^{2,1}$ regularity theory fails then there cannot be a $C^{2,1}$ Schauder estimate.

Similarly, there exists $f \in C^0(B_1(0))$ and a solution $u \in C^2(B_1(0) \setminus \{0\})$ to $\Delta u = f$ in $B_1(0)$ such that $u \notin C^2$ in any neighborhood of the origin and thus it is generally false that we have a C^2 Schauder estimate of the form

$$|u|'_{2;B_{R/2}(x_0)} \leq C \left(|u|_{0;B_R(x_0)} + R^2 |f|'_{0;B_R(x_0)} \right).$$

(2) The Schauder estimate bounds the $C^{2,\mu}$ norm of u on a smaller ball in terms of norms on a larger ball. It is generally false that

$$|u|'_{2,\mu;B_R(x_0)} \leq C \left(|u|_{0;B_R(x_0)} + R^2 |f|'_{0,\mu;B_R(x_0)} \right); \tag{7}$$

for example, if $u(x_1 + ix_2) = \mathrm{Re}((x_1 + ix_2)^k)$ for $(x_1, x_2) \in B_1(0)$ and a large integer $k \geq 2$, then u is harmonic and it is easy to check that

$$\sup_{B_1(0)} |u| = 1, \quad \sup_{B_1(0)} |D^2 u| = \sqrt{2} k^2, \quad \sup_{B_{1/2}(0)} |D^2 u| = \sqrt{2} k^2 (1/2)^k,$$

so that (7) is clearly false but (6) holds true.

(3) The Schauder estimate is invariant under scaling, i.e. transformation of $x \mapsto y + \rho x$ for $y \in \mathbb{R}^n$ and $\rho > 0$. This is an important quality to have in an estimate, as we can always rescale a solution to an elliptic equation to find another solution to another elliptic equation and thus our estimates should be invariant, or at least well behaved, under scaling. To see this scale invariance, let $\widetilde{u}(x) = u(x_0 + Rx)$. It is easy to check that

$$\widetilde{u}(x) = u(x_0 + Rx), \quad D\widetilde{u}(x) = RDu(x_0 + Rx), \quad D^2\widetilde{u}(x) = R^2 D^2 u(x_0 + Rx). \tag{8}$$

Hence it follows that $\widetilde{u}$ satisfies

$$a^{ij}(x_0 + Rx) D_{ij} \widetilde{u}(x) + Rb^i(x_0 + Rx) D_i \widetilde{u}(x) + R^2 c(x_0 + Rx) \widetilde{u}(x) = R^2 f(x_0 + Rx). \tag{9}$$

Let $\widetilde{f}(x) = R^2 f(x_0 + Rx)$. Observe that by (8)

$$\begin{aligned}
[D^2 \widetilde{u}]_{\mu;B_1(0)} &= \inf_{x \neq y \in B_1(0)} \frac{|R^2 D^2 u(x_0 + Rx) - R^2 D^2 u(x_0 + Ry)|}{|x-y|^\mu} \\
&= \inf_{x \neq y \in B_1(0)} \frac{|R^2 D^2 u(x_0 + Rx) - R^2 D^2 u(x_0 + Ry)|}{|(x_0 + Rx) - (x_0 + Ry)|^\mu} \cdot R^\mu \\
&= R^{2+\mu} [D^2 u]_{\mu;B_R(x_0)},
\end{aligned}$$

so by (8)

$$|\widetilde{u}|_{2,\mu;B_1(0)} = |u|'_{2,\mu;B_R(x_0)} \tag{10}$$

and similarly

$$\begin{aligned}
|a^{ij}(x_0 + Rx)|_{0,\mu;B_1(0)} &= |a^{ij}|'_{0,\mu;B_R(x_0)}, \quad |Rb^i(x_0 + Rx)|_{0,\mu;B_1(0)} = R|b^i|'_{0,\mu;B_R(x_0)}, \\
|R^2 c(x_0 + Rx)|_{0,\mu;B_1(0)} &= R^2 |c|'_{0,\mu;B_R(x_0)}, \quad |\widetilde{f}|_{0,\mu;B_1(0)} = R^2 |f|'_{0,\mu;B_R(x_0)}.
\end{aligned} \tag{11}$$

By (9), (10), and (11), (6) is equivalent to

$$|\widetilde{u}|_{2,\mu;B_{1/2}(0)} \leq C \left(|\widetilde{u}|_{0;B_1(0)} + |\widetilde{f}|_{0,\mu;B_1(0)} \right).$$

Now let's prove the Schauder estimate. We will do this by a contradiction argument involving scaling. This argument has several steps.

Step 1: First we claim that to prove (6), it suffices to prove under the hypotheses of the Schauder lemma that for every constant $\delta > 0$ there is a constant $C = C(\delta, n, \mu, \lambda, \Lambda) \in (0, \infty)$

$$R^{2+\mu} [D^2 u]_{\mu;B_{R/2}(x_0)} \leq \delta R^{2+\mu} [D^2 u]_{\mu;B_R(x_0)} + C \left(|u|'_{2;B_R(x_0)} + R^2 |f|'_{0,\mu;B_R(x_0)} \right). \tag{12}$$

Suppose that we knew (12) held true. Then by interpolation, for every $\varepsilon > 0$,

$$|u|'_{2;B_R(x_0)} \leq \varepsilon [D^2 u]_{2,\mu;B_R(x_0)} + C(n, \mu, \varepsilon) |u|_{0;B_R(x_0)},$$

so by taking $\varepsilon = \delta / C$ for C in (12), (12) implies that

$$R^{2+\mu} [D^2 u]_{\mu;B_{R/2}(x_0)} \leq 2\delta R^{2+\mu} [D^2 u]_{\mu;B_R(x_0)} + C \left(|u|_{0;B_R(x_0)} + R^2 |f|'_{0,\mu;B_R(x_0)} \right) \tag{13}$$

for some constant $C = C(\delta, n, \mu, \lambda, \Lambda) \in (0, \infty)$. Note that (13) will hold if we replace $B_R(x_0)$ with any ball contained in $B_R(x_0)$, so by the Simple Abstract Lemma, upon choosing $\delta(n) > 0$,

$$R^{2+\mu} [D^2 u]_{\mu;B_{R/2}(x_0)} \leq C \left(|u|_{0;B_R(x_0)} + R^2 |f|'_{0,\mu;B_R(x_0)} \right) \tag{14}$$

for some constant $C = C(n, \mu, \lambda, \Lambda) \in (0, \infty)$ as required. By interpolation

$$|u|_{2,\mu;B_{R/2}(x_0)} \leq |u|_{0;B_{R/2}(x_0)} + C(n, \mu) R^{2+\mu} [D^2 u]_{\mu;B_{R/2}(x_0)})$$

for some constant $C = C(n, \mu, \lambda, \Lambda) \in (0, \infty)$, so (14) implies

$$|u|_{2,\mu;B_{R/2}(x_0)} \leq C \left(|u|_{0;B_R(x_0)} + R^2|f|'_{0,\mu;B_R(x_0)}\right)$$

for some constant $C = C(n, \mu, \lambda, \Lambda) \in (0, \infty)$.

By translating and rescaling, we can assume without loss of generality that $x_0 = 0$ and $R = 1$ so that in place of (12) we want to prove that for every constant $\delta > 0$ there is a constant $C = C(\delta, n, \mu, \lambda, \Lambda) \in (0, \infty)$

$$[D^2u]_{\mu;B_{1/2}(0)} \leq \delta[D^2u]_{\mu;B_1(0)} + C(|u|_{2;B_1(0)} + |f|_{0,\mu;B_1(0)}). \tag{15}$$

Step 2: We shall prove (15) by contradiction. Suppose that for some sequence of $u_k \in C^2(\overline{B_1(0)})$ and $a_k^{ij}, f_k \in C^2(\overline{B_1(0)})$,

$$a_k^{ij}D_{ij}u_k + b_k^iD_iu_k + cu_k = f_k \text{ in } B_1(0), \tag{16}$$

and

$$\lambda|\xi|^2 \leq a_k^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \text{ for } x \in B_1(0),\ \xi \in \mathbb{R}^n, \tag{17}$$

$$\sum_{i,j=1}^n \|a_k^{ij}\|_{C^{0,\mu}(B_1(0))} + \sum_{i=1}^n \|b_k^i\|_{C^{0,\mu}(B_1(0))} + \|c_k\|_{C^{0,\mu}(B_1(0))} \leq \beta, \tag{18}$$

where $\lambda, \Lambda, \beta \in (0, \infty)$ are fixed constants, but for some fixed $\delta > 0$,

$$[D^2u_k]_{\mu;B_{1/2}(0)} > \delta[D^2u_k]_{\mu;B_1(0)} + k\left(|u_k|_{2;B_1(0)} + |f|_{0,\mu;B_1(0)}\right). \tag{19}$$

Select distinct $x_k, y_k \in B_{1/2}(0)$ such that

$$\frac{|D^2u_k(x_k) - D^2u_k(y_k)|}{|x_k - y_k|^\mu} > \frac{1}{2}[D^2u_k]_{\mu;B_{1/2}(0)} \tag{20}$$

and let $\rho_k = |x_k - y_k|$. Observe that

$$\begin{aligned}
\frac{1}{2}[D^2u_k]_{\mu;B_{1/2}(0)} &< \frac{|D^2u_k(x_k) - D^2u_k(y_k)|}{|x_k - y_k|^\mu} && \text{by (20)} \\
&\leq \frac{2|D^2u|_{0;B_1(0)}}{\rho_k^\mu} \\
&\leq \frac{2}{k\rho_k^\mu}[D^2u_k]_{\mu;B_{1/2}(0)} && \text{by (19),}
\end{aligned}$$

so $\rho_k^\mu \leq 4/k$ and thus $\rho_k \to 0$ as $k \to \infty$.

Step 3: Now we are going to rescale and "blow-up". Define

$$\widetilde{u}_k(x) = \frac{u_k(x_k + \rho_kx) - u_k(x_k) - \rho_k\sum_{i=1}^n D_iu_k(x_k)x_i - (1/2)\rho_k^2\sum_{i,j=1}^n D_{ij}u_k(x_k)x_ix_j}{\rho_k^{2+\mu}[D^2u_k]_{\mu;B_1(0)}},$$

$$\widetilde{a}_k^{ij}(x) = a_k^{ij}(x_k + \rho_kx), \quad \widetilde{b}_k^i(x) = \rho_kb_k^i(x_k + \rho_kx), \quad \widetilde{c}_k(x) = \rho_k^2c_k(x_k + \rho_kx),$$

$$\widetilde{f}_k(x) = \frac{f_k(x_k + \rho_kx) - f_k(x_k)}{\rho_k^\mu[D^2u_k]_{\mu;B_1(0)}},$$

$$\xi_k = \frac{y_k - x_k}{\rho_k}.$$

Clearly $\widetilde{u}_k(0) = 0$, $D\widetilde{u}_k(0) = 0$, $D^2\widetilde{u}_k(0) = 0$, $[D^2\widetilde{u}_k]_{B_{1/2\rho_k}(0)} \leq 1$, and $|\xi_k| = 1$. By (19) and (20),

$$|D^2\widetilde{u}_k(\xi_k) - D^2\widetilde{u}_k(0)| > \frac{\delta}{2}. \tag{21}$$

Now we want to let $k \to \infty$. By Bolzano-Weierstrass, after passing to a subsequence, ξ_k converges to some ξ with $|\xi| = 1$. We know that $[D^2\widetilde{u}_k]_{\mu;B_{1/2\rho_k}(0)} \leq 1$.

Since $D^2\widetilde{u}_k(0) = 0$,

$$|D^2\widetilde{u}_k(x)| = |D^2\widetilde{u}_k(x) - D^2\widetilde{u}_k(0)| \leq 1 \cdot |x|^\mu \leq \sigma^\mu$$

for all $x \in B_\sigma(0)$ and $\sigma \in (0, \infty)$ provided k sufficiently large (how large k has to be off course depends on σ) and similarly

$$|\widetilde{u}_k(x)| \leq \sigma^{2+\mu}, \quad |D\widetilde{u}_k(x)| \leq \sigma^{1+\mu},$$

for all $x \in B_\sigma(0)$ and $\sigma \in (0, \infty)$ provided k sufficiently large. Therefore by Arzela-Ascoli, $\widetilde{u}_k$ converges to some function $\widetilde{u}$ in C^2 on compact subsets of $\mathbb{R}^n$. Note that again by the properties of $\widetilde{u}_k$, in particular (21),

$$[D^2\widetilde{u}]_{\mu;\mathbb{R}^n} \leq 1, \quad |D^2\widetilde{u}(\xi) - D^2\widetilde{u}(0)| \geq \delta/2. \tag{22}$$

By (18),

$$\sup_{B_{1/2\rho_k}(0)} |\widetilde{a}_k^{ij}| \leq \sup_{B_1(0)} |a_k^{ij}| \leq \beta,$$
$$[\widetilde{a}_k^{ij}]_{\mu;B_\sigma(0)} = \sup_{x \neq y \in B_\sigma(0)} \frac{|a_k^{ij}(x_k + \rho_k x) - a_k^{ij}(x_k + \rho_k y)|}{|x - y|^\mu} \leq \rho_k^\mu [a_k^{ij}]_{\mu;B_1(0)},$$

so by Arzela-Ascoli, $\widetilde{a}_k^{ij}$ converges to some function $\widetilde{a}^{ij}$ uniformly on compact subsets of $\mathbb{R}^n$. Moreover, $[\widetilde{a}^{ij}]_{\mu;\mathbb{R}^n} = 0$, so $\widetilde{a}^{ij}$ is in fact constant. By (18),

$$\lambda|\xi|^2 \leq \widetilde{a}^{ij}\xi_i\xi_j \leq \Lambda|\xi|^2 \text{ for } \xi \in \mathbb{R}^n.$$

By (18),

$$\sup_{B_{1/2\rho_k}(0)} |\widetilde{b}_k^i| \leq \rho_k \sup_{B_1(0)} |b_k^i| \leq \beta\rho_k,$$
$$\sup_{B_{1/2\rho_k}(0)} |\widetilde{c}_k^i| \leq \rho_k^2 \sup_{B_1(0)} |c_k| \leq \beta\rho_k^2,$$

so $\widetilde{b}_k^i \to 0$ and $\widetilde{c}_k \to 0$ uniformly on compact subsets of $\mathbb{R}^n$.

By (19),

$$\begin{aligned}
\sup_{B_\sigma(0)} |\widetilde{f}| &= \sup_{B_\sigma(0)} \frac{|f_k(x_k + \rho_k x) - f_k(x_k)|}{\rho_k^\mu [D^2 u_k]_{\mu;B_1(0)}} \\
&\leq \frac{[f_k]_{\mu;B_1(0)} (\rho_k \sigma)^\mu}{\rho_k^\mu [D^2 u_k]_{\mu;B_1(0)}} \\
&\leq \frac{\sigma^\mu}{k},
\end{aligned}$$

so $\widetilde{f}_k \to 0$ uniformly on compact subsets of $\mathbb{R}^n$.

Now we want to show that $\widetilde{u}_k$ satisfies an elliptic equation. By rescaling (16) by $x \mapsto x_k + \rho_k x$ and dividing by $\rho_k^{2+\mu}[D^2 u_k]_{\mu;B_{1/2}(0)}$,

$$\begin{aligned}
&\widetilde{a}_k^{ij} D_{ij}\widetilde{u}_k + \widetilde{b}_k^i D_i \widetilde{u}_k + \widetilde{c}\widetilde{u}_k \\
&= \frac{a_k^{ij}(x_k+\rho_k x) D_{ij} u_k(x_k+\rho_k x) + b_k^i(x_k+\rho_k x) D_i u_k(x_k+\rho_k x) + c_k(x_k+\rho_k x) u_k(x_k+\rho_k x)}{\rho_k^{\mu}[D^2 u_k]_{\mu;B_1(0)}} \\
&\quad - \frac{a_k^{ij}(x_k+\rho_k x) D_{ij} u(x_k) + b_k^i(x_k+\rho_k x) D_i u(x_k) + \rho_k b_k^i(x_k+\rho_k x) D_{ij} u(x_k) x_j}{\rho_k^{\mu}[D^2 u_k]_{\mu;B_1(0)}} \\
&\quad - \frac{c_k(x_k+\rho_k x) u(x_k) + \rho_k c_k(x_k+\rho_k x) D_i u(x_k) x_i + (1/2)\rho_k^2 c_k(x_k+\rho_k x) D_{ij} u(x_k) x_i x_j}{\rho_k^{\mu}[D^2 u_k]_{\mu;B_1(0)}} \\
&= \frac{f_k(x_k+\rho_k x) - f_k(x_k)}{\rho_k^{\mu}[D^2 u_k]_{\mu;B_1(0)}} - \frac{(a_k^{ij}(x_k+\rho_k x) - a_k^{ij}(x_k)) D_{ij} u(x_k)}{\rho_k^{\mu}[D^2 u_k]_{\mu;B_1(0)}} \\
&\quad - \frac{(b_k^i(x_k+\rho_k x) - b_k^i(x_k)) D_i u(x_k) + (c_k(x_k+\rho_k x) - c_k(x_k)) u_k(x_k)}{\rho_k^{\mu}[D^2 u_k]_{\mu;B_1(0)}} \\
&\quad - \frac{\rho_k b_k^i(x_k+\rho_k x) D_{ij} u(x_k) x_j + \rho_k c_k(x_k+\rho_k x) D_i u(x_k) x_i + (1/2)\rho_k^2 c_k(x_k+\rho_k x) D_{ij} u(x_k) x_i x_j}{\rho_k^{\mu}[D^2 u_k]_{\mu;B_1(0)}}
\end{aligned}$$

in $B_{1/2\rho_k}(0)$, hence

$$\left| \widetilde{a}_k^{ij} D_{ij}\widetilde{u}_k + \widetilde{b}_k^i D_i \widetilde{u}_k + \widetilde{c}_k \widetilde{u}_k - \widetilde{f}_k \right| \leq C(n,\sigma)\frac{\beta}{k} \tag{23}$$

on $B_\sigma(0)$ for all $\sigma \in (0,\infty)$ and k sufficiently large. (Note that it might be helpful to consider this computation in the special case that $b^i = 0$ and $c = 0$ on $B_1(0)$. We then compute

$$\begin{aligned}
\widetilde{a}_k^{ij} D_{ij}\widetilde{u}_k &= \frac{a_k^{ij}(x_k+\rho_k x) D_{ij} u_k(x_k+\rho_k x) - a_k^{ij}(x_k+\rho_k x) D_{ij} u(x_k)}{\rho_k^{\mu}[D^2 u_k]_{\mu;B_1(0)}} \\
&= \frac{a_k^{ij}(x_k+\rho_k x) D_{ij} u_k(x_k+\rho_k x) - a_k^{ij}(x_k) D_{ij} u(x_k) - (a_k^{ij}(x_k+\rho_k x) - a_k^{ij}(x_k)) D_{ij} u(x_k)}{\rho_k^{\mu}[D^2 u_k]_{\mu;B_1(0)}} \\
&= \frac{f_k(x_k+\rho_k x) - f_k(x_k)}{\rho_k^{\mu}[D^2 u_k]_{\mu;B_1(0)}} - \frac{(a_k^{ij}(x_k+\rho_k x) - a_k^{ij}(x_k)) D_{ij} u(x_k)}{\rho_k^{\mu}[D^2 u_k]_{\mu;B_1(0)}}
\end{aligned}$$

in $B_{1/2\rho_k}(0)$, where

$$\frac{|a_k^{ij}(x_k+\rho_k x) - a_k^{ij}(x_k)||D_{ij} u(x_k)|}{\rho_k^{\mu}[D^2 u_k]_{\mu;B_1(0)}} = \frac{|a_k^{ij}(x_k+\rho_k x) - a_k^{ij}(x_k)|}{\rho_k^{\mu}} \cdot + \frac{|D_{ij} u(x_k)|}{[D^2 u_k]_{\mu;B_1(0)}} \leq \beta |x|^{\mu} \cdot \frac{1}{k}.\Bigg)$$

By letting $k \to \infty$ in (23),

$$\widetilde{a}^{ij} D_{ij} \widetilde{u} = 0 \text{ in } \mathbb{R}^n. \tag{24}$$

Step 4: Now we have a solution $\widetilde{u}$ to the elliptic equation (24) for some constant $\widetilde{a}^{ij}$ satisfying (22), which recall states that

$$[D^2 \widetilde{u}]_{\mu;\mathbb{R}^n} \leq 1, \quad |D^2 \widetilde{u}(\xi) - D^2 \widetilde{u}(0)| \geq \delta/2.$$

Recall from the maximum principle lectures, after an orthogonal change of coordinates we may take (24) to have the form

$$\sum_{i=1}^{n} \lambda_i D_{ii} \widetilde{u} = 0 \text{ in } \mathbb{R}^n$$

for some constants $\lambda_i > 0$ and (21) is unchanged. Let

$$w(x_1, x_2, \ldots, x_n) = \widetilde{u}(\sqrt{\lambda_1} x_1, \sqrt{\lambda_2} x_2, \ldots, \sqrt{\lambda_n} x_n)$$

so that w is harmonic on $\mathbb{R}^n$ and by (21), $[D^2 w]_{\mu;\mathbb{R}^n} < \infty$ and $D^2 w$ is not constant on $\mathbb{R}^n$. Since w is a C^2 harmonic function on $\mathbb{R}^n$, w is smooth on $\mathbb{R}^n$. Hence $D^2 w$ is a harmonic function on $\mathbb{R}^n$ such that its Hölder coefficient on $\mathbb{R}^n$ is finite and it is not constant on $\mathbb{R}^n$, contradicting the Liousville lemma (see below).

Lemma 3 (Liousville lemma)**.** *There is no non-constant harmonic function u on $\mathbb{R}^n$ with $[u]_{\mu;\mathbb{R}^n} < \infty$.*

Proof. Recall that if u is harmonic then each derivative $D_i u$ is also harmonic. Thus for every $y \in \mathbb{R}^n$ and $R > 0$,

$$\begin{aligned}
|D_i u(y)| &= \left| \frac{1}{\omega_n R^n} \int_{B_R(y)} D_i u \right| && \text{(by the mean value property)} \\
&= \left| \frac{1}{\omega_n R^n} \int_{\partial B_R(y)} u(x) \frac{x_i}{|x|} dx \right| && \text{(by the divergence theorem)} \\
&\leq \frac{1}{\omega_n R^n} \int_{\partial B_R(y)} |u(x)| dx \\
&\leq \frac{1}{\omega_n R^n} \int_{\partial B_R(y)} (|u(y)| + |u(x) - u(y)|) dx \\
&\leq \frac{1}{\omega_n R^n} \int_{\partial B_R(y)} (|u(y)| + [u]_{\mu;\mathbb{R}^n} |x - y|^\mu dx && \text{(since } [u]_{\mu;\mathbb{R}^n} < \infty) \\
&\leq n R^{-1} |u(y)| + n R^{\mu - 1} [u]_{\mu;\mathbb{R}^n},
\end{aligned}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. By letting $R \to \infty$ we obtain $D_i u(y) = 0$ for all $y \in \mathbb{R}^n$ and $i = 1, 2, \ldots, n$, i.e. u is constant. □

Theorem 1 (Interior Schauder estimates)**.** *Let $\mu \in (0, 1)$ and $\Omega' \subset\subset \Omega$ be bounded domains in $\mathbb{R}^n$. Suppose $u \in C^{2,\mu}(\Omega) \cap C^0(\overline{\Omega})$ solves the uniformly elliptic equation*

$$Lu = a^{ij} D_{ij} u + b^i D_i u + cu = f \text{ in } \Omega,$$

where the coefficients $a^{ij}, b^i, c \in C^{0,\mu}(\overline{\Omega})$ satisfy

$$\begin{aligned}
&\lambda |\xi|^2 \leq a^{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2 \text{ for } x \in \Omega,\ \xi \in \mathbb{R}^n, \\
&|a^{ij}|_{0,\mu;\Omega} + |b^i|_{0,\mu;\Omega} + |c|_{0,\mu;\Omega} \leq \beta,
\end{aligned}$$

for some constants $\lambda, \Lambda, \beta > 0$ and $f \in C^{0,\mu}(\overline{\Omega})$. Then

$$|u|_{2,\mu;\Omega'} \leq C \left(|u|_{0;\Omega} + |f|_{0,\mu;\Omega} \right)$$

for some constant $C = C(n, \mu, \lambda, \Lambda, \beta, \Omega', \Omega) \in (0, \infty)$.

Proof. Let $d = \mathrm{dist}(\Omega', \partial\Omega)$. Given $x \in \Omega'$, $B_d(x) \subset \Omega$, so by the interior Schauder estimate proved last time,

$$\begin{aligned} |u(x)| + (d/2)|Du(x)| + (d/2)^2|D^2u(x)| + (d/2)^{2+\mu}[D^2u]_{\mu;B_{d/2}(x)} &\leq |u|_{2;B_{d/2}(x)} \\ &\leq C\left(|u|_{0;\Omega} + d^2|f|_{0,\mu;\Omega}\right) \end{aligned} \tag{25}$$

for some constant $C = C(n, \mu, \lambda, \Lambda, \beta) \in (0, \infty)$, giving us the required bounds on u, Du, and D^2u. Suppose $x, y \in \Omega$ with $x \neq y$. If $|x - y| < d/2$, then by (25),

$$(d/2)^{2+\mu}\frac{|D^2u(x) - D^2u(y)|}{|x-y|^\mu} \leq (d/2)^{2+\mu}[D^2u]_{\mu;B_d(y)} \leq C\left(|u|_{0;\Omega} + d^2|f|_{0,\mu;\Omega}\right).$$

If instead $|x - y| \geq d/2$, by the bound on D^2u in (25),

$$(d/2)^{2+\mu}\frac{|D^2u(x) - D^2u(y)|}{|x-y|^\mu} \leq 2(d/2)^2|D^2u|_{0;B_{d/2}(x)} \leq 2C\left(|u|_{0;\Omega} + d^2|f|_{0,\mu;\Omega}\right).$$

□